

NONHOMOGENEOUS, CONTINUOUS-TIME MARKOV CHAINS DEFINED BY SERIES OF PROPORTIONAL INTENSITY MATRICES

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Let A_1, A_2, \dots , be commuting intensity matrices of homogeneous, continuous-time Markov chains. The irreducibility and ergodicity of nonhomogeneous, continuous-time Markov chains defined by intensity matrices of the form $Q(t) = \sum h_n(t)A_n$, $h_n(t) \geq 0$, are studied in terms of corresponding discrete-time chains. By defining transition matrices of homogeneous, discrete-time chains as $\bar{P}_n = I + (1/a_n)A_n$ it is found that if one P_n is irreducible and the corresponding $h_n(t)$ does not vanish then $Q(t)$ is irreducible. Similarly, if one of the P_n 's (or the average of a finite number of the P_n 's) is ergodic and the corresponding $h_n(t)$ is large enough ($\int_x^\infty h_n(t) du = \infty$) then the nonhomogeneous, continuous-time chain is ergodic. For an intensity matrix A and a nonnegative function $h(t)$ with $h(t)\|A\| < 1$ for all t , it is shown that $Q(t) = \sum_{n=1}^x (h(t))'' A^n$ is an intensity matrix. Moreover, if $P = I + (1/a)A$ is ergodic and if $\int_x^\infty h(u) du = \infty$, then $Q(t)$ is ergodic.

1. Introduction and definitions

Let $\{X(t)\}_{t \geq 0}$ be a continuous-time Markov chain on a countable state space S with transition matrices $P(s, t)$ where $P_{ij}(s, t) = P\{X(t) = j | X(s) = i\}$ for every $t \geq s \geq 0$ and $i, j \in S$. For standard chains, the intensity matrices can be defined as

$$Q(t) = \lim_{\substack{h+k \rightarrow 0^+ \\ (h=0 \text{ or } k=0)}} \frac{P(t-h, t+k) - I}{n+k}.$$

Under certain restrictions, one can define the Markov chain using the intensity matrices. We assume that the chain is regular so that there is only one Markov process defined by $\{Q(t)\}$. The chains considered are conservative and have no instantaneous states: $-\infty < q_{ii}(t) \leq 0$, $q_{ij}(t) \geq 0$ for $i \neq j$ and $\sum_{j \in S} q_{ij}(t) = 0$. Also, $P_{ij}(s, t) \geq 0$ and $\sum_{j \in S} P_{ij}(s, t) = 1$. (For details see [1] or [3].)

For a time homogeneous process, $P(s, t) = P(t-s)$ and $Q(t) = Q$ for all $t \geq 0$.

The long-run properties of continuous-time, homogeneous Markov chains are often studied in terms of their intensity matrices. One technique was introduced by

Yong [9] who defined $\bar{P} = I + Q/q$ in order to discuss the ergodicity of the continuous time chain with intensity matrix Q in terms of the discrete-time chain defined by the transition matrix \bar{P} .

Nonhomogeneous, continuous-time chains are more difficult to study, but several specific cases have been considered. Yong's technique was used by Scott and Isaacson [8] to discuss a class of nonhomogeneous chains defined by proportional intensities, $Q(t) = h(t)Q$. Johnson [6] used it on constant causative chains, $Q(t) = tC + Q$ where C and Q are commuting intensity matrices of two homogeneous, continuous-time chains. Here we extend this technique to consider a larger subclass of non-homogeneous, continuous-time chains defined by intensity matrices $\sum_{n=1}^N h_n(t)A_n$. To ensure that each A_n is regular, it is assumed that $\|A_n\| < a_n < \infty$ (see [2]). Each $h_n(t)$ is continuous and nonnegative so that $h_n(t)A_n$ is an intensity matrix. In order to ensure the convergence of the series $Q(t) = \sum_{n=1}^{\infty} h_n(t)A_n$, it is assumed that there exist $b_n \geq 0$ such that $\sum_{n=1}^{\infty} b_n < \infty$ and $\sup_{t \geq 0} \|h_n(t)A_n\| \leq b_n$. As Lemma 1.2, below, indicates, this condition ensures that the long-run distribution will be the same for any of the homogeneous chains which happen to be ergodic. $X(t)$ will be used to designate chains with $Q(t) = \sum_{n=1}^{\infty} h_n(t)A_n$ which satisfy the above restrictions.

Since the zero matrix is an intensity matrix which commutes with all other matrices, $Q(t) = \sum_{n=1}^{\infty} h_n(t)A_n$ can represent a finite sum. In the case $Q(t) = \sum_{n=1}^N h_n(t)A_n$, the restriction that the functions $h_n(t)$ be bounded could be discarded. One such case is that of the constant causative chains, where $Q(t) = tA_1 + A_2$, which are considered by Johnson [5, 6].

As an example consider the TIAA/CREF university retirement program which has three different funds: the CREF stock fund, the CREF money market fund, and the TIAA fund. The rules are that one can move money from the CREF stock or money market funds to one another or to the TIAA fund, but no funds can be moved out of the TIAA fund. With this in mind set

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -a & a \\ 0 & b & -b \end{pmatrix}.$$

Then $A_1A_2 = A_2A_1$. The size of the function h will vary seasonally depending on economic conditions.

Under the above restrictions it is not difficult to see that the transition matrix is given by

$$P(s, t) = \exp \left\{ \sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right\}.$$

Since each A is assumed to be bounded in norm by some a_n , the matrices \bar{P}_n defined by

$$\bar{P}_n = I + (1/a_n)A_n$$

are stochastic and can be considered the transition matrices of homogeneous,

discrete-time Markov chains. $P(s, t)$ can be rewritten as

$$P(s, t) = \exp \left\{ - \sum_{n=1}^{\infty} a_n \int_s^t h_n(u) du \right\} \exp \left\{ \sum_{n=1}^{\infty} a_n \int_s^t h_n(u) du \bar{P}_n \right\}. \quad (1)$$

These chains are used to study the irreducibility and ergodicity of $X(t)$.

Definition. A Markov chain is irreducible if for any $i, j \in S$ and $s \geq 0$ there exist $t_1 > s$ and $t_2 > s$ such that $p_{ij}(s, t_1) > 0$ and $p_{ji}(s, t_2) > 0$.

Definition. A Markov chain is ergodic if there exists a collection $\{\pi_j\}_{j \in S}$ such that $\pi_j \geq 0$ for every $j \in \mathcal{J}$, $\sum_{j \in \mathcal{J}} \pi_j = 1$ and for any $i \in \mathcal{J}$ and any $s \geq 0$, $\lim_{t \rightarrow \infty} p_{ij}(s, t) = \pi_j$ for every $j \in \mathcal{J}$.

The definition of ergodicity involves only the convergence of the elements of the transition matrices $P(s, t)$. Because of the stochastic nature of $P(s, t)$ and π , this can be strengthened to the following (see [4]):

Theorem 1.1. Suppose $\pi = \{\pi_j\}_{j \in \mathcal{J}}$ exists and satisfies $\pi_j \geq 0$ for every $j \in \mathcal{J}$ and $\sum_{j \in \mathcal{J}} \pi_j = 1$. A Markov chain with transition matrices $P(s, t)$ is ergodic with limit π if and only if for any $i \in \mathcal{J}$ and for any $s \geq 0$, $\lim_{t \rightarrow \infty} \sum_{j \in \mathcal{J}} |p_{ij}(s, t) - \pi_j| = 0$. \square

A stronger form of ergodicity is defined using the norm $\|A\| = \sup_{i \in \mathcal{J}} \{\sum_{j \in \mathcal{J}} |a_{ij}|\}$.

Definition. A Markov chain with transition matrix $P(s, t)$ is strongly ergodic if there exists a row constant stochastic matrix L with rows $\pi = \{\pi_j\}_{j \in \mathcal{J}}$ such that for every $s \geq 0$,

$$\lim_{t \rightarrow \infty} \|P(s, t) - L\| = 0.$$

Clearly, strong ergodicity implies ergodicity.

One preliminary result is given for future use. For a proof see [6].

Lemma 1.2. Suppose P is the transition matrix of an ergodic, discrete-time, homogeneous Markov chain with stationary distribution π . If R is a transition matrix of another discrete-time, homogeneous Markov chain, not necessarily ergodic, and R commutes with P , then $\pi R = \pi$. \square

2. Irreducibility

If any of the \bar{P}_n 's represents an irreducible, homogeneous Markov chain and the corresponding $h_n(t)$ does not vanish then $X(t)$ is irreducible. A more general statement is the following.

Theorem 2.1. *If there exists a finite set $W = \{n_1, n_2, \dots, n_m\}$ of positive integers such that $(1/m) \sum_{n \in W} \bar{P}_n$ is the transition matrix of an irreducible Markov chain and if for every $s \geq 0$ there exists numbers $t_n = t_n(s)$ such that $\int_s^{t_n} h_n(u) du > 0$ for each $n \in W$, then $X(t)$ is irreducible.*

Proof. Fix $s \geq 0$ and $i, j \in S$.

Since $h_n(t) \geq 0$ for each n , $\int_s^t h_n(u) du$ is increasing in t . By setting $T > \max_{n \in W} \{t_n\}$ it follows that $\int_s^T h_n(u) du > 0$ for each $n \in W$. Suppose $a_{n_0} \int_s^T h_{n_0}(u) du = \min_{n \in W} \{a_n \int_s^T h_n(u) du\} > 0$. By (1),

$$\begin{aligned} p_{ij}(s, T) &= \exp \left\{ - \sum_{n=1}^{\infty} a_n \int_s^T h_n(u) du \right\} \\ &\quad \cdot \exp \left\{ \sum_{n \notin W} a_n \int_s^T h_n(u) du \bar{P}_n + \sum_{n \in W} a_n \int_s^T h_n(u) du \bar{P}_n \right\}_{ij} \\ &= \exp \left\{ - \sum_{n=1}^{\infty} a_n \int_s^T h_n(u) du \right\} \\ &\quad \cdot \sum_{k \in \mathcal{J}} \exp \left\{ \sum_{n \notin W} a_n \int_s^T h_n(u) du \bar{P}_n \right. \\ &\quad \quad \left. + \sum_{n \in W} \left(a_n \int_s^T h_n(u) du - a_{n_0} \int_s^T h_{n_0}(u) du \right) \bar{P}_n \right\}_{ik} \\ &\quad \cdot \exp \left\{ \sum_{n \in W} a_{n_0} \int_s^T h_{n_0}(u) du \bar{P}_n \right\}_{kj}. \end{aligned}$$

Letting δ_{ik} be the Kronecker delta this implies

$$p_{ij}(s, T) \geq \exp \left\{ - \sum_{n=1}^{\infty} a_n \int_s^T h_n(u) du \right\} \sum_{k \in \mathcal{J}} \delta_{ik} \exp \left\{ a_{n_0} \int_s^T h_{n_0}(u) du \sum_{n \in W} \bar{P}_n \right\}_{kj}.$$

Since $(1/m) \sum_{n \in W} \bar{P}_n$ is irreducible, there exists a positive integer L such that $(\sum_{n \in W} \bar{P}_n)_{ij}^L > 0$. Thus,

$$p_{ij}(s, T) \geq \exp \left\{ - \sum_{n=1}^{\infty} a_n \int_s^T h_n(u) du \right\} \frac{1}{L!} \left(a_{n_0} \int_s^T h_{n_0}(u) du \right)^L \left(\sum_{n \in W} \bar{P}_n \right)_{ij}^L > 0. \quad \square$$

Note. As noted earlier, homogeneous chains, chains with proportional intensities and constant causative chains are all special cases. In these cases Theorem 2.1 reduces to known results. A homogeneous chain defined by $Q(t) = A$ is irreducible if $\bar{P} = I + (1/a)A$ is the transition matrix of an irreducible, discrete-time Markov chain. (See [9].) Scott [7], showed that for proportional intensities ($Q(t) = h(t)A$, where $h(t) \geq 0$, $\lim_{t \rightarrow \infty} h(t) = 1$, $H(t) = \int_0^t h(s) ds$ is strictly increasing and $H(t) \rightarrow \infty$ as $t \rightarrow \infty$) irreducibility was equivalent to that of the discrete-time chain defined by

$\bar{P} = I + (1/a)A$. The constant causative chain defined by $Q(t) = tA_1 + A_2$ is irreducible if $\bar{P}_1 = I + (1/a_1)A_1$, $\bar{P}_2 = I + (1/a_2)A_2$ or $\frac{1}{2}(\bar{P}_1 + \bar{P}_2)$ is irreducible. (See [6].)

3. Ergodicity

The ergodicity of one \bar{P}_n is sufficient for the ergodicity of $X(t)$ if $h_n(t)$ remains large for large t .

Theorem 3.1. For $n \geq 1$, suppose each P_n is a stochastic matrix and $f_n(s, t) \geq 0$. Assume further that $\sum_{n=1}^{\infty} f_n(s, t) < \infty$ and the P_k 's commute. Let

$$P(s, t) = \exp\left\{-\sum_{n=1}^{\infty} f_n(s, t)\right\} \exp\left\{\sum_{n=1}^{\infty} f_n(s, t) P_n\right\}.$$

If the discrete-time, homogeneous chain with transition matrix P_1 is ergodic with limit π and if for any $s \geq 0$, $\lim_{t \rightarrow \infty} f_1(s, t) = \infty$, then $P(s, t)$ is ergodic with limit π .

Note that by Lemma 1.2 all the P_n 's have the same invariant π .

Proof. Since the P_n 's commute, for fixed $i, j \in \mathcal{S}$,

$$p_{ij}(s, t) = \exp\{-f_1(s, t)\} \sum_{k \in \mathcal{S}} \left[\sum_{m=0}^{\infty} \frac{1}{m!} (f_1(s, t))^m (P_1)_{ik}^m \cdot \exp\left\{-\sum_{n=2}^{\infty} f_n(s, t)\right\} \exp\left\{\sum_{n=2}^{\infty} f_n(s, t) P_n\right\}_{kj} \right].$$

Let

$$R(s, t) = \exp\left\{-\sum_{n=2}^{\infty} f_n(s, t)\right\} \exp\left\{\sum_{n=2}^{\infty} f_n(s, t) P_n\right\}.$$

It can be shown that $R(s, t)$ is stochastic. Since each P_n commutes with P_1 , $R(s, t)$ commutes with P_1 . By Lemma 1.2, $\pi R(s, t) = \pi$. Therefore,

$$\begin{aligned} & \exp\{-f_1(s, t)\} \sum_{k \in \mathcal{S}} \left(\sum_{m=0}^{\infty} \frac{1}{m!} (f_1(s, t))^m ((P_1)_{ik}^m - \pi_k) \right) r_{kj}(s, t) \\ &= p_{ij}(s, t) - \exp\{-f_1(s, t)\} \exp\{f_1(s, t)\} \sum_{k \in \mathcal{S}} \pi_k r_{kj}(s, t) \\ &= p_{ij}(s, t) - \pi_j. \end{aligned}$$

Choose $\varepsilon > 0$. By Theorem 1.1, there exists an N such that for $m > N$,

$$\sum_{k \in \mathcal{S}} |(P_1)_{ik}^m - \pi_k| < \frac{\varepsilon}{2}.$$

Thus

$$\begin{aligned} |p_{ij}(s, t) - \pi_j| &\leq \exp\{-f_1(s, t)\} \sum_{k \in \mathcal{I}'} \left(\sum_{m=0}^{\infty} \frac{1}{m!} (f_1(s, t))^m |(P_1)_{ik}^m - \pi_k| \right) \cdot 1 \\ &= \exp\{-f_1(s, t)\} \left\{ \sum_{m=0}^{N-1} \frac{1}{m!} (f_1(s, t))^m \sum_{k \in \mathcal{I}'} |(P_1)_{ik}^m - \pi_k| \right. \\ &\quad \left. + \sum_{m=N}^{\infty} \frac{1}{m!} (f_1(s, t))^m \sum_{k \in \mathcal{I}'} |(P_1)_{ik}^m - \pi_k| \right\}. \end{aligned}$$

Since $\lim_{t \rightarrow \infty} f_1(s, t) = \infty$, for $0 \leq m \leq N-1$, one can find a T such that for $t > T$,

$$\exp\{-f_1(s, t)\} \frac{1}{m!} (f_1(s, t))^m < \frac{\varepsilon}{4N}.$$

Therefore for $t \geq T$,

$$|p_{ij}(s, t) - \pi_j| \leq 2 \sum_{m=0}^{N-1} \frac{\varepsilon}{4N} + \frac{\varepsilon}{2} = \varepsilon. \quad \square$$

In the case where $X(t)$ is defined by $Q(t) = \sum_{n=1}^{\infty} h_n(t) A_n$, the functions $f_n(s, t)$ are replaced by $\int_s^t h_n(u) du$. The assumption that one of the \bar{P}_n 's is ergodic can be replaced by the ergodicity of a linear combination of the \bar{P}_n 's.

Corollary 3.2. Suppose there exists a finite set $W = \{n_1, n_2, \dots, n_3\}$ such that the chain which has transition matrix $(1/m) \sum_{n \in W} \bar{P}_n$ is ergodic with limit π . If for each $s \geq 0$ and each $n \in W$ $\int_s^{\infty} h_n(u) du = \infty$, then $X(t)$ is ergodic with limit π .

Proof. Let $f(s, t) = \min_{n \in W} \{a_n \int_s^t h_n(u) du\}$, then $mf(s, t) \geq 0$ and $\lim_{t \rightarrow \infty} mf(s, t) = \infty$ for each s . Define

$$f_n(s, t) = \begin{cases} a_n \int_s^t h_n(u) du - f(s, t) & \text{for } n \in W, \\ a_n \int_s^t h_n(u) du & \text{for } n \notin W. \end{cases}$$

Then $f_n(s, t) \geq 0$ and $\sum_{n=1}^{\infty} a_n \int_s^t h_n(u) du = \sum_{i=1}^m f(s, t) + \sum_{n=1}^{\infty} f_n(s, t)$.

Using (1),

$$\begin{aligned} P(s, t) &= \exp\left\{-\sum_{n=1}^{\infty} a_n \int_s^t h_n(u) du\right\} \exp\left\{\sum_{n=1}^{\infty} a_n \int_s^t h_n(u) du \bar{P}_n\right\} \\ &= \exp\left\{-\sum_{i=1}^m f(s, t) - \sum_{n=1}^{\infty} f_n(s, t)\right\} \\ &\quad \cdot \exp\left\{\sum_{n \in W} f(s, t) \bar{P}_n + \sum_{n=1}^{\infty} f_n(s, t) \bar{P}_n\right\} \\ &= \exp\left\{-mf(s, t) - \sum_{n=1}^{\infty} f_n(s, t)\right\} \\ &\quad \cdot \exp\left\{mf(s, t) \left(\frac{1}{m} \sum_{n \in W} \bar{P}_n\right) + \sum_{n=1}^{\infty} f_n(s, t) \bar{P}_n\right\}. \end{aligned}$$

Since the P_n 's commute with each other, they commute with $(1/m) \sum_{n \in W} \bar{P}_n$. Thus the hypotheses of the theorem hold and the corollary follows. \square

This result can be extended to strong ergodicity with little substantial change in the proof.

Theorem 3.3. *If there exists a finite set $W = \{n_1, n_2, \dots, n_m\}$ of positive integers such that $(1/m) \sum_{n \in W} \bar{P}_n$ is strongly ergodic with limit L and for each $n \in W$ $\int_0^\infty h_n(u) du = \infty$, then $X(t)$ is strongly ergodic with limit L . \square*

Note. For the special cases of homogeneous, proportional intensity and constant causative chains, the following results are related. Yong [9] compares geometric ergodicity of the homogeneous chains defined by \bar{P} and A , $\bar{P} = I + (1/a)A$. Scott [7] extended this result to strong ergodicity: If the discrete-time chain with transition matrix \bar{P} is strongly ergodic, then so is the continuous-time chain with intensity matrix A . In the case of proportional intensities, $Q(t) = h(t)A$, the corresponding result is found in [8]. If \bar{P} is strongly ergodic and $\int_s^t h(s) ds \rightarrow \infty$ as $t - s \rightarrow \infty$, then the continuous time chain is strongly ergodic. For the constant causative chain defined by $Q(t) = tA_1 + A_2$, if either \bar{P}_1 or \bar{P}_2 is ergodic (strongly ergodic) the $Q(t)$ is also ergodic (strongly ergodic). (See [6].)

The fact that $\int_0^\infty h_n(u) du = \infty$ is quite important for the ergodicity of $X(t)$, as seen in the following theorem.

Theorem 3.4. *Suppose $X(t)$ has intensity matrix given by the finite sum $Q(t) = \sum_{k=1}^n h_k(t)A_k$. If for each $k = 1, 2, \dots, n$, $\int_0^\infty h_k(u) du < \infty$, then $X(t)$ is not ergodic.*

Note. In this case, $X(t)$ will also not be strongly ergodic.

Proof. $P(s, t) = \exp(Q(s, t))$ where $Q(s, t) = \sum_k A_k \int_s^t h_k(u) du$. Then $\|Q(s, t)\| \leq \sum \|A_k\| \int_s^t h_k(u) du \rightarrow 0$ as $s, t \uparrow \infty$. Thus

$$\|P(s, t) - I\| \leq \sum_{k=1}^{\infty} \frac{1}{k!} \|Q(s, t)\|^k = e^{\|Q(s, t)\|} - 1 \rightarrow 0$$

as $s, t \rightarrow \infty$ and the result follows. \square

4. Geometric series

Because of the difficulty in finding a sequence of matrices which commute, it is tempting to try to replace the sequence of intensity matrices $\{A_n\}_{n=1}^\infty$ by the sequence $\{A^n\}_{n=1}^\infty$, the powers of A . These powers clearly commute with each other. The

nonhomogeneous chain might then be defined by $Q(t) = \sum_{n=1}^{\infty} h_n(t)A^n$. Unfortunately, while A is an intensity matrix, A^n need not be one. $Q(t)$ defined in this way may not be an intensity matrix and therefore may not define a Markov chain. The following example is a case where this happens.

Example. Let

$$A = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}, \quad A^2 = (-3)A.$$

Let

$$h_n(t) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ (\frac{1}{6})^n & \text{if } n \text{ is even.} \end{cases}$$

Notice that $h_n(t)\|A^n\| \leq (\frac{1}{6})^n 4^n = (\frac{2}{3})^n$.

$$\begin{aligned} Q(t) &= \sum_{n=1}^{\infty} h_n(t)A^n = \sum_{n=1}^{\infty} (\frac{1}{6})^{2n}(-3)^{2n-1}A \\ &= \begin{pmatrix} \frac{1}{9} & -\frac{1}{9} \\ -\frac{2}{9} & \frac{2}{9} \end{pmatrix} \end{aligned}$$

which is not an intensity matrix.

However, if one also assumes $h_n(t) = (h(t))^n$, then $\sum_{n=1}^{\infty} (h(t)A)^n$ is an intensity matrix.

Theorem 4.1. Suppose A is an intensity matrix and $h(t)$ is nonnegative function. If $h(t)\|A\| < 1$ for every t , then

$$Q(t) = \sum_{n=1}^{\infty} (h(t))^n A^n$$

is an intensity matrix.

Proof. For fixed t , let $B = h(t)A$. Define $D = \frac{1}{2}(I + B)$.

Since $\|D\| \leq \frac{1}{2}(\|I\| + \|B\|) < 1$, $\sum_{n=1}^{\infty} D^n$ converges.

$$\begin{aligned} \sum_{n=1}^{\infty} D^n &= \sum_{n=1}^{\infty} (\frac{1}{2})^n \sum_{k=0}^n \binom{n}{k} B^k \\ &= \sum_{n=1}^{\infty} (\frac{1}{2})^n \binom{n}{0} B^0 + \sum_{n=1}^{\infty} \sum_{k=1}^n (\frac{1}{2})^n \binom{n}{k} B^k \\ &= I(1) + \sum_{k=1}^{\infty} \frac{(\frac{1}{2})^k B^k}{k!} \sum_{n=k}^{\infty} (\frac{1}{2})^{n-k} \frac{n!}{(n-k)!}. \end{aligned}$$

Since for $|x| < 1$, $(1-x)^{-1} = \sum_{n=0}^{\infty} x^n$, taking k derivatives on both sides gives

$$k!(1-x)^{-k-1} = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} x^{n-k}.$$

Thus

$$\sum_{n=1}^{\infty} D^n = I + \sum_{k=1}^{\infty} \frac{(\frac{1}{2})^k B^k}{k!} k! (\frac{1}{2})^{-k-1} = I + 2 \sum_{k=1}^{\infty} B^k.$$

Therefore,

$$Q(t) = \sum_{k=1}^{\infty} B^k = \frac{1}{2} \left(\sum_{n=1}^{\infty} D^n - I \right). \quad (2)$$

By the properties of intensity matrices,

$$0 \leq b_{ij} \leq -b_{ii} \leq \frac{1}{2} \|B\| < \frac{1}{2}.$$

Hence, each element of D is nonnegative. From (2), the off-diagonal elements of $Q(t)$ are clearly nonnegative.

By induction on n it can be shown that

$$\sum_{j \in I'} (D^n)_{ij} = (\frac{1}{2})^n.$$

Therefore by (2),

$$q_{ii}(t) = \frac{1}{2} \left(\sum_{n=1}^{\infty} (D^n)_{ii} - 1 \right) \leq \frac{1}{2} \sum_{n=1}^{\infty} (\frac{1}{2})^n - \frac{1}{2} = 0$$

and

$$\sum_{j \in I'} q_{ij}(t) = \sum_{j \in I'} \frac{1}{2} \left(\sum_{n=1}^{\infty} (D^n)_{ij} - \delta_{ij} \right) = 0. \quad \square$$

It can be shown that when $Q(t) = \sum_{n=1}^{\infty} (h(t))^n A^n$,

$$P(s, t) = \exp \left\{ \sum_{n=1}^{\infty} \int_s^t (h(u))^n du A^n \right\}.$$

One might guess, in light of the results found for $Q(t) = \sum_{n=1}^{\infty} h_n(t) A_n$, that if $\bar{P} = I + A/a$ is ergodic and $\int_s^{\infty} h(u) du = \infty$ for every $s \geq 0$ then the chain defined by $Q(t) = \sum_{n=1}^{\infty} (h(t))^n A^n$ is also ergodic. This is in fact the case.

Lemma 4.2. *If $Q(t) = \sum_{n=1}^{\infty} (h(t))^n A^n$ where $h(t) \|A\| < 1$ for any $t \geq 0$ then one can choose $a \geq 0$ so that $\bar{P} = I + A/a$ is stochastic and*

$$P(s, t) = \exp \left\{ - \sum_{k=1}^{\infty} \int_s^t \frac{(ah(u))^k}{(1+ah(u))^{k+1}} du \right\} \exp \left\{ \sum_{k=1}^{\infty} \int_s^t \frac{(ah(u))^k}{(1+ah(u))^{k+1}} du \bar{P}^k \right\}.$$

Proof. By Theorem 4.1, $Q(t)$ is an intensity matrix. Since $h(t)\|A\| < 1$, one can choose a so that $\sup_{i \in \mathcal{S}} \{ |a_{ii}| \} < a$ and $h(t)a < 1$ for any $t \geq 0$. Since $A = a(\bar{P} - I)$,

$$Q(t) = \sum_{n=1}^{\infty} (h(t))^n a^n (\bar{P} - I)^n$$

which can be shown to be

$$Q(t) = \sum_{k=1}^{\infty} \frac{(ah(t))^k}{(1+ah(t))^{k+1}} \bar{P}^k - \left(\frac{ah(t)}{1+ah(t)} \right) I.$$

But $0 \leq ah(t) < 1$ implies that $1 \geq 1/(1+ah(t)) > \frac{1}{2}$. Thus,

$$1 > ah(t) \geq \frac{ah(t)}{1+ah(t)} \geq \frac{1}{2} ah(t) \geq 0.$$

Therefore,

$$\sum_{k=1}^{\infty} \frac{(ah(t))^k}{(1+ah(t))^{k+1}} = \frac{ah(t)}{1+ah(t)}.$$

Hence,

$$Q(t) = \sum_{k=1}^{\infty} \frac{(ah(t))^k}{(1+ah(t))^{k+1}} \bar{P}^k - \sum_{k=1}^{\infty} \frac{(ah(t))^k}{(1+ah(t))^{k+1}} I.$$

Therefore,

$$\begin{aligned} P(s, t) &= \exp \left\{ \int_s^t Q(u) du \right\} \\ &= \exp \left\{ - \sum_{k=1}^{\infty} \int_s^t \frac{(ah(u))^k}{(1+ah(u))^{k+1}} du \right\} \\ &\quad \cdot \exp \left\{ \sum_{k=1}^{\infty} \int_s^t \frac{(ah(u))^k}{(1+ah(u))^{k+1}} du \bar{P}^k \right\}. \quad \square \end{aligned}$$

Theorem 4.3. Suppose $Q(t) = \sum_{n=1}^{\infty} (h(t))^n A^n$ where $h(t)\|A\| < 1$. If \bar{P} (as found in Lemma 4.2) is ergodic and $\int_0^{\infty} h(u) du = \infty$, then the nonhomogeneous chain defined by $Q(t)$ is ergodic.

Proof. Since $0 \leq ah(u) < 1$, $1 \geq 1/(1+ah(u))^2 \geq \frac{1}{4}$. Thus,

$$\int_0^{\infty} \frac{ah(u)}{(1+ah(u))^2} du \geq \frac{1}{4} \int_0^{\infty} ah(u) du = \infty.$$

The proof follows from Theorem 3.1. \square

These results can be extended to strong ergodicity.

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